Parameter estimation and control for a class of systems with nonlinear parametrization

Ivan Tyukin*, Danil Prokhorov, Cees van Leeuwen

Abstract

We propose novel parameter estimation algorithms for a class of dynamical systems with nonlinear parametrization. The class is initially restricted to smooth monotonic functions with respect to a linear functional of the parameters. We show that under this restriction standard persistent excitation suffices to ensure exponentially fast convergence of the estimates to the actual values of unknown parameters. Subsequently, our approach is extended to cases in which the monotonicity assumption holds only locally. We show that excitation with high-frequency of oscillations is sufficient to ensure convergence. Two practically relevant examples are given in order to illustrate the effectiveness of the approach.

Keywords: nonlinear parametrization, parameter estimation, persistent excitation, exponential convergence, monotonic functions

Corresponding author:

Ivan Tyukin

Laboratory for Perceptual Dynamics,

RIKEN Brain Science Institute,

2-1, Hirosawa, Wako-shi, Saitama,

351-0198, Japan

phone: +81-48-462-1111 extension 7436

fax: +81-48-467-7236

e-mail: tyukinivan@brain.riken.jp

1 Introduction

Broad areas of applied and fundamental science require parameter identification of nonlinear systems. Research of these systems has made substantial progress in identification of both static and dynamic linearly parameterized systems [17],[11], [3], as well as static nonlinear ones [5],[13],[29],[16]. Parameter estimation of dynamic systems with nonlinear parametrization, however, has long remained an open issue. One possible way, in principle, to address this problem is to try to derive the estimator for the general nonlinear case.

^{*}Laboratory for Perceptual Dynamics, RIKEN (Institute for Physical and Chemical Research) Brain Science Institute, 2-1, Hirosawa, Wako-shi, Saitama, 351-0198, Japan, e-mail: {tyukinivan}@brain.riken.jp

[†]Ford Research Laboratory, Dearborn, MI, 48121, USA, e-mail: dprokhor@ford.com

[‡]Laboratory for Perceptual Dynamics, RIKEN (Institute for Physical and Chemical Research) Brain Science Institute, 2-1, Hirosawa, Wako-shi, Saitama, 351-0198, Japan, e-mail: {ceesvl}@brain.riken.jp

In this hard problem a breakthrough has resulted in an advanced method [7]. This method applies to a large class of nonlinear systems. Despite this major advancement the price for such generality are several theoretical and practical limitations. First, it is required that uncertainty be Lipshitz in time. Second, extra control is needed in order to dominate the nonlinearity during identification. The third and most important restriction is the necessity to satisfy the *nonlinear persistent excitation condition*, which computationally is more difficult to check than conventional persistent excitation assumptions [18], [19]¹ and often is not easy to satisfy if state-dependent nonlinearities are allowed.

An alternative strategy would be to consider a class of nonlinear parameterizations that is narrower, but still sufficiently broad to be practically relevant. Currently there are a number of models, for instance Hammerstain (Wiener) models [20],[23],[12],[2], with specific restrictions on the nonlinearity in the parameters that allow to avoid the problems arising in the general parametrization case. These models, however, handle only static input (output) nonlinearities. Local linear (nonlinear) model techniques [15], [28], [10] constitute another promising tool. These models, on the other hand, are not always physically plausible. In practice in order to identify parameters of actual physical processes in a system, it is often necessary with these models to refit the data to the original nonlinearly parameterized model.

In this article we address the problem of parameter estimation of dynamic nonlinear parameterized systems in a way that compromises between the pros and cons of both strategies mentioned above. In particular, we allow nonlinear state-dependent parametrization in the model, while restricting the nonlinearities in parameters to be of a certain practically relevant class. As a result we obtain parameter estimation procedures that are not limited to Lipshitz nonlinearities in time. These procedures neither require domination of the nonlinearity, nor do they rely on nonlinear persistent excitation conditions. On the other hand, the class of nonlinear parameterizations that we propose is wide enough to include a variety of models in physics, mechanics, physiology and neural computation [1], [21], [4], [9]. For this new class of parameterizations we show that conventional persistent excitation conditions guarantee exponential convergence of the estimates to the actual values of the parameters. Moreover, in case our assumptions are satisfied only locally, sufficiently high frequency of excitation still ensures convergence.

The paper is organized as follows. In Section 2 we formulate the problem, Section 3 contains the main results of the paper, in Section 3 we provide two practically relevant illustrative applications of our method, and Section 4 concludes the paper.

¹See also [22], [31], where relaxed formulations of persistent excitation conditions are discussed.

2 Problem Formulation

Let the following system be given:

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}) + \mathbf{g}_1(\mathbf{x})u,
\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}, \boldsymbol{\theta}) + \mathbf{g}_2(\mathbf{x})u, \tag{1}$$

where

$$\mathbf{x}_{1} = (x_{11}, \dots, x_{1m_{1}})^{T} \in \mathbb{R}^{m_{1}}$$

$$\mathbf{x}_{2} = (x_{21}, \dots, x_{2m_{2}})^{T} \in \mathbb{R}^{m_{2}}$$

$$\mathbf{x} = (x_{11}, \dots, x_{1m_{1}}, x_{21}, \dots, x_{2m_{2}})^{T} \in \mathbb{R}^{n}$$

 $\theta \in \Omega_{\theta} \in \mathbb{R}^d$ is a vector of unknown parameters, u is the control input, and functions $\mathbf{f}_1 : \mathbb{R}^n \to \mathbb{R}^{m_1}$, $\mathbf{f}_2 : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{m_2}$, $\mathbf{g}_1 : \mathbb{R}^n \to \mathbb{R}^{m_1}$, $\mathbf{g}_2 : \mathbb{R}^n \to \mathbb{R}^{m_2}$ are locally bounded². Vector $\mathbf{x} \in \mathbb{R}^n$ is a state vector, and vectors \mathbf{x}_1 , \mathbf{x}_2 are referred to as uncertainty independent and uncertainty dependent partitions of \mathbf{x} respectively. We assume that Ω_{θ} is bounded, and therefore without loss of generality it is safe to assume that Ω_{θ} is a closed ball or hypercube in \mathbb{R}^d .

For the sake of compactness we introduce the following alternative description for (1):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}) + \mathbf{g}(\mathbf{x})u,\tag{2}$$

where

$$\mathbf{g}(\mathbf{x}) = (g_{11}(\mathbf{x}), \dots, g_{1m_1}(\mathbf{x}), g_{21}(\mathbf{x}), \dots, g_{2m_2}(\mathbf{x}))^T$$
$$\mathbf{f}(\mathbf{x}) = (f_{11}(\mathbf{x}), \dots, f_{1m_1}(\mathbf{x}), f_{21}(\mathbf{x}, \boldsymbol{\theta}), \dots, f_{2m_2}(\mathbf{x}, \boldsymbol{\theta}))^T$$

Our goal is to derive both the control function $u(\mathbf{x},t)$ and estimator $\hat{\boldsymbol{\theta}}(t)$ such that all trajectories of the system are bounded and the estimate $\hat{\boldsymbol{\theta}}(t)$ converges to unknown $\boldsymbol{\theta} \in \Omega_{\theta}$ asymptotically. In addition, in order to ensure boundedness of the trajectories we will restrict all possible motions of system (2) to an admissible domain in the system state space. As a measure of closeness of the trajectories to the desired solution we introduce the smooth error function $\psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \ \psi \in C^1$. The function $\psi(\mathbf{x},t)$ is bounded in t for every bounded \mathbf{x} . The target manifold, therefore, is given by

$$\psi(\mathbf{x},t) = 0$$

Consider the transverse dynamics of system (2) with respect to $\psi(\mathbf{x},t)$:

$$\dot{\psi} = L_{\mathbf{f}(\mathbf{x},\boldsymbol{\theta})}\psi(\mathbf{x},t) + L_{\mathbf{g}(\mathbf{x})}\psi(\mathbf{x},t)u + \frac{\partial\psi(\mathbf{x},t)}{\partial t},\tag{3}$$

²Function $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be locally bounded if for any $\|\mathbf{x}\| < \delta$ there exists constant $D(\delta) > 0$ such that the following holds: $\|\mathbf{f}(\mathbf{x})\| \le D(\delta)$.

where $L_{\mathbf{f}(\mathbf{x},\boldsymbol{\theta})}$ is Lie derivative of function $\psi(\mathbf{x},t)$ with respect to vector field $\mathbf{f}(\mathbf{x},\boldsymbol{\theta})$. Let us further assume that $L_{\mathbf{g}(\mathbf{x})}\psi(\mathbf{x},t)$ is separated from zero (i.e. there exists positive $\delta > 0$ such that $|L_{\mathbf{g}(\mathbf{x})}\psi(\mathbf{x},t)| > \delta$ for any $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}_+$). This assumption automatically implies existence of the inverse $L_{\mathbf{g}(\mathbf{x})}\psi(\mathbf{x},t)^{-1}$. Hence, we can select control input u from the following class of functions:

$$u(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) = (L_{\mathbf{g}(\mathbf{x})} \psi(\mathbf{x}, t))^{-1} (-L_{\mathbf{f}(\mathbf{x}, \hat{\boldsymbol{\theta}})} \psi(\mathbf{x}, t) - \varphi(\psi) - \frac{\partial \psi(\mathbf{x}, t)}{\partial t}), \tag{4}$$

where

$$\varphi : \mathbb{R} \to \mathbb{R}, \ \varphi(\psi) \in C^1, \ \varphi(\psi)\psi > 0 \ \forall \ \psi \neq 0, \ \lim_{\psi \to \infty} \int_0^{\psi} \varphi(\xi) d\xi = \infty.$$
 (5)

Denoting $L_{\mathbf{f}(\mathbf{x},\boldsymbol{\theta})}\psi(\mathbf{x},t) = f(\mathbf{x},\boldsymbol{\theta},t)$ and taking into account (4) we can rewrite equation (3) in the following manner:

$$\dot{\psi} = f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - \varphi(\psi) \tag{6}$$

It is natural to require that boundedness of $\psi(\mathbf{x},t)$ implies boundedness of the state insofar as $\psi(\mathbf{x},t)$ stands for the deviation from target manifold $\psi(\mathbf{x},t) = 0$. Let us formally introduce this requirement in the following assumption:

Assumption 1 For the given function $\psi(\mathbf{x},t)$ the following holds:

$$\psi(\mathbf{x},t) \in L_{\infty} \Rightarrow \mathbf{x} \in L_{\infty}$$

Assumption 1 can be considered a bounded input - bounded state assumption for system (1) along the constraint $\psi(\mathbf{x},t) = v(t)$, where functions u is chosen to satisfy this requirement and signal v(t) serves as input. If, however, boundedness of the state is not required or is achieved by extra control, Assumption 1 can be removed from the statements of our results or replaced, when necessary, with the requirement for the function $f(\mathbf{x}, \boldsymbol{\theta}, t)$ to be globally bounded in \mathbf{x} and locally bounded in $\boldsymbol{\theta}$.

So far the only deviation from standard descriptions of the problem resides in the specification of function $f(\mathbf{x}, \boldsymbol{\theta}, t)$. Since a general parametrization of function $f(\mathbf{x}, \boldsymbol{\theta}, t)$ is methodologically difficult to deal with but solutions provided for the restricted classes of nonlinearities often yield physically implausible models, we have opted to search for a new class of practically reasonable parameterizations. Such a class should be able to include a sufficiently broad range of physical models, in particular those with nonlinear parametrization; they should also, in principle, be able to handle arbitrary (in the class of smooth functions) nonlinearity in states. As a candidate for such a parametrization we suggest nonlinear functions that satisfy the following assumption:

Assumption 2 (Monotonicity and Linear Growth Rate in Parameters) There exists function $\alpha(\mathbf{x},t)$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^d$ and D > 0 such that

$$(f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t))(\boldsymbol{\alpha}(\mathbf{x}, t)^T(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})) > 0 \ \forall \ f(\mathbf{x}, \boldsymbol{\theta}, t) \neq f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)$$

$$|f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t)| \le D|\boldsymbol{\alpha}(\mathbf{x}, t)^T (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})|, D > 0$$

The first statement in Assumption 2 holds, for example, for every smooth nonlinear function which is monotonic with respect to a linear functional over a vector of parameters $\boldsymbol{\theta}$: $f(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x})^T \boldsymbol{\theta}, t)$. The second inequality is satisfied if the function $f(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x})^T \boldsymbol{\theta}, t)$ does not grow faster than a linear function in variable $\boldsymbol{\phi}(\mathbf{x})^T \boldsymbol{\theta}$ for every $\mathbf{x} \in \mathbb{R}^n$. This set of conditions naturally extends systems that are linear in parameters to those with nonlinear parametrization. In addition to linearly parameterized systems, Assumption 2 covers a considerable large variety of practically relevant models with nonlinear parametrization. These include effects of stiction forces [1], slip and surface dependent friction given by the "magic formula" [21], smooth saturation, and dead-zones in mechanical systems. The set of functions covered by Assumption 2 further includes nonlinearities in models of bio-reactors [4]. The class of functions $f(\mathbf{x}, \boldsymbol{\theta}, t)$ specified in Assumption 2 can also serve as nonlinear replacement of the functions that are linear in their parameters in a variety of piecewise approximation models. Last but not least it includes sigmoid and Gaussian nonlinearities, which are favored in neuro and fuzzy control and mathematical models of neural processes [9].

In this article we attempt to resolve the following main issue: how to design the estimator $\hat{\boldsymbol{\theta}}(\mathbf{x},t)$ which ensures convergence of the estimates to the actual values of a-priori unknown parameter $\boldsymbol{\theta}$, and what further restrictions (if any) on functions $f(\mathbf{x}, \boldsymbol{\theta}, t)$ are to be satisfied in order to guarantee such convergence?

3 Main Results

Let us introduce the following adaptation algorithm ³:

$$\hat{\boldsymbol{\theta}}(\mathbf{x},t) = \Gamma(\hat{\boldsymbol{\theta}}_{P}(\mathbf{x},t) + \hat{\boldsymbol{\theta}}_{I}(t));$$

$$\hat{\boldsymbol{\theta}}_{P}(\mathbf{x},t) = \psi(\mathbf{x},t)\boldsymbol{\alpha}(\mathbf{x},t) - \Psi(\mathbf{x},t)$$

$$\dot{\hat{\boldsymbol{\theta}}}_{I} = \varphi(\psi(\mathbf{x},t))\boldsymbol{\alpha}(\mathbf{x},t) + \partial\Psi(\mathbf{x},t)/\partial t - \psi(\mathbf{x},t)(\partial\boldsymbol{\alpha}(\mathbf{x},t)/\partial t) - (\psi(\mathbf{x},t)L_{\mathbf{f}_{1}}\boldsymbol{\alpha}(\mathbf{x},t) - L_{\mathbf{f}_{1}}\Psi(\mathbf{x},t)) - (\psi(\mathbf{x},t)L_{\mathbf{g}_{1}}\boldsymbol{\alpha}(\mathbf{x},t) - L_{\mathbf{g}_{1}}\Psi(\mathbf{x},t))u(\mathbf{x},\hat{\boldsymbol{\theta}},t)$$

$$+\beta(\mathbf{x},t)(\mathbf{f}_{2}(\mathbf{x},\hat{\boldsymbol{\theta}}) + \mathbf{g}_{2}(\mathbf{x})u(\mathbf{x},\hat{\boldsymbol{\theta}},t)), \tag{7}$$

where functions $\Psi(\mathbf{x},t)$, $\beta(\mathbf{x},t)$ satisfy the following condition with respect to the vector-fields of system (1) and function $\alpha(\mathbf{x},t)$:

Assumption 3 There exists function $\Psi(\mathbf{x},t)$ such that

$$\frac{\partial \Psi(\mathbf{x},t)}{\partial \mathbf{x}_2} - \psi(\mathbf{x},t) \frac{\partial \alpha(\mathbf{x},t)}{\partial \mathbf{x}_2} = \beta(\mathbf{x},t)$$

where $\beta(\mathbf{x},t)$ is ether zero or, if $\mathbf{f}_2(\mathbf{x},\boldsymbol{\theta})$ is differentiable in $\boldsymbol{\theta}$, satisfies the following:

$$\beta(\mathbf{x},t)\mathcal{F}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\theta}') \geq 0 \ \forall \boldsymbol{\theta},\boldsymbol{\theta}' \in \Omega_{\boldsymbol{\theta}}, \ \mathbf{x} \in \mathbb{R}^n$$

³Parameter adjustment algorithms (7) can be considered as generalizations of the algorithms introduced earlier by the authors in [26],[25]. In these works we analyzed stabilizing properties of these algorithms in connection with realizability issues. Parameter convergence and identifying properties of algorithms (7) were not addressed there.

$$\mathcal{F}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\theta}') = \int_0^1 \frac{\partial \mathbf{f}_2(\mathbf{x}, \mathbf{s}(\lambda))}{\partial \mathbf{s}} d\lambda, \quad \mathbf{s}(\lambda) = \boldsymbol{\theta}' \lambda + \boldsymbol{\theta} (1 - \lambda)$$

Assumption 3 can be viewed as a kind of a structural restriction. Indeed, one can easily see that it automatically holds for the cases where $\frac{\partial \alpha(\mathbf{x},t)}{\partial \mathbf{x}_2} = 0$, i.e. when function $\alpha(\mathbf{x},t)$ does not depend explicitly on vector \mathbf{x}_2 , which stands for the uncertainty-dependent partition of system (1). Assumption 3 holds also for one-dimensional uncertainty-dependent partitions if function $\psi(\mathbf{x},t)\frac{\partial \alpha(\mathbf{x},t)}{\mathbf{x}_2}$ is Riemann-integrable with respect to \mathbf{x}_2 (vector \mathbf{x}_2 is one-dimensional in this case, $\beta(\mathbf{x},t)=0$). Although it may seem to be difficult to find functions $\Psi(\mathbf{x},t)$ satisfying requirements of Assumption 3, in general the difficulty of the problem can significantly be reduced by *embedding* the system dynamics into one of a higher order, for which Assumption 3 is satisfied a-priori. Sufficient conditions ensuring existence of such embedding for the parameterizations of general structure are provided in [26]. For systems where parametric uncertainty can be reduced to vector fields with low-triangular structure the embedding is given in [27]. An alternative and the easiest way to construct this embedding is to design a system of which the output $\hat{\mathbf{x}}_2(t)$ tracks vector \mathbf{x}_2 with the prescribed level of performance by use of high-gain robust observers. The former is then used in the adjustment algorithm as replacement for the latter. This makes it possible to reduce the problem either to one of already considered cases of independence of $\alpha(\mathbf{x},t)$ on uncertainty-dependent partitions or to single-dimension partitions of \mathbf{x}_2 . This technique is illustrated in detail in the examples section.

Properties of system (1) with control (4) and adaptation algorithm (7) are summarized in Theorem 1 and Theorem 2.

Theorem 1 (Stability and Convergence) Let system (1), (4), (7) be given and Assumptions 2-3 hold.

Then

P1)
$$\varphi(\psi(t)) \in L_2, \ \dot{\psi}(t) \in L_2$$
:

P2) $\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(t)\|_{\Gamma^{-1}}^2$ is non-increasing;

P3)
$$f((\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}(t), t)) \in L_2$$
.

Furthermore,

$$\|\varphi(\psi)\|_{2}^{2} \leq 2Q(\psi) + \|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|_{(2D\Gamma)^{-1}}^{2}, \quad \|\dot{\psi}\|_{2}^{2} \leq 2Q(\psi) + \|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|_{(2D\Gamma)^{-1}}^{2}$$

$$\|\psi\|_{\infty} \leq \Lambda \left(Q(\psi) + \|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|_{(4D\Gamma)^{-1}}^{2}\right), \tag{8}$$

where $Q(\psi) = \int_0^{\psi(\mathbf{x}(0),0)} \varphi(\varsigma) d\varsigma$ and $\Lambda(d) = \max_{|\psi|} \{|\psi| \mid \int_0^{|\psi|} \varphi(\varsigma) d\varsigma = d\}.$

If Assumption 1 is satisfied and function $f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)$ is locally bounded with respect to \mathbf{x} , $\hat{\boldsymbol{\theta}}$ and uniformly bounded with respect to t, then

P4) trajectories of the system are bounded and $\psi(\mathbf{x}(t)) \to 0$ as $t \to \infty$;

If in addition functions φ , $f(\mathbf{x}, \boldsymbol{\theta}, t) \in C^1$, derivative $\partial f(\mathbf{x}, \boldsymbol{\theta}, t)/\partial t$ is uniformly bounded in t, function $\alpha(\mathbf{x}, t)$ is locally bounded with respect to \mathbf{x} and uniformly bounded with respect to t, then

P5)
$$\dot{\psi} \to 0$$
 as $t \to \infty$; $f((\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}(t), t)) \to 0$ as $t \to \infty$.

Proofs of Theorem 1 and subsequent results are given in the Appendix.

Theorem 1 ensures for algorithms (7) asymptotic reaching of the control goal and boundedness of the solutions of the closed-loop system. In addition, it provides improved transient performance, which can be characterized by a-priori computable L_2 norms for $\dot{\psi}$ and ψ . In the case where $f(\mathbf{x}, \boldsymbol{\theta}, t) \neq f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)$ along the system solutions it further guarantees reduction of parametric uncertainties (property P2).

So far we have assumed that functions $\varphi(\psi)$ may vary freely in the class of functions specified by condition (5). It is possible, however, to show that the transient performance of system (1) with algorithms (7) can further be improved when functions $\varphi(\psi)$ are linear in ψ . An additional assumption on the growth rate of function $f(\mathbf{x}, \boldsymbol{\theta})$ in $\boldsymbol{\theta}$ will make the whole system exponentially stable. This new assumption is formulated as follows:

Assumption 4 For the given function $f(\mathbf{x}, \boldsymbol{\theta})$ in (6) and function $\alpha(\mathbf{x}, t)$, satisfying Assumption 2, there exists a positive constant $D_1 > 0$ such that

$$|f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t)| \ge D_1 |\boldsymbol{\alpha}(\mathbf{x}, t)^T (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})|$$

Assumption 4 extends Assumption 2 by stipulating a lower bound for the growth rate of nonlinear function $f(\mathbf{x}, \boldsymbol{\theta})$ in $\boldsymbol{\theta}$. This assumption allows us to show that exponential convergence of $\hat{\boldsymbol{\theta}}$ to $\boldsymbol{\theta}$ will automatically result in exponentially fast convergence of function $\psi(\mathbf{x}, t)$ to the origin. Furthermore, it ensures exponential convergence of $\hat{\boldsymbol{\theta}}$ to $\boldsymbol{\theta}$ for any positive-definite constant Γ . These results are formulated in the following theorem:

Theorem 2 (Exponential Convergence) Let Assumptions 2-3 hold and $\varphi(\psi) = K\psi$, K > 0. Then P6) function $\psi(\mathbf{x}(t),t)$ converges exponentially fast into the domain $|\psi(\mathbf{x}(t),t)| \leq 0.5\sqrt{\|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|_{(KD\Gamma)^{-1}}^2}$. Specifically, the following holds: $|\psi(\mathbf{x}(t),t)| \leq |\psi(\mathbf{x}(0),0)|e^{-Kt} + 0.5\sqrt{\|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|_{(KD\Gamma)^{-1}}^2}$

Furthermore, let Assumption 1 hold, function $\alpha(\mathbf{x},t)$ be locally bounded with respect to \mathbf{x} and uniformly bounded in t; for any bounded \mathbf{x} there exist $D_1 > 0$ such that $|f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t)| \ge D_1 |\alpha(\mathbf{x}, t)^T (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})|$, function $\alpha(\mathbf{x}, t)$ is persistently exciting:

$$\exists L > 0, \ \delta > 0: \ \int_{t}^{t+L} \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau) \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau)^{T} d\tau \ge \delta I \ \forall t > 0,$$
 (9)

where $I \in \mathbb{R}^{d \times d}$ – identity matrix. Then

P7) both $\psi(\mathbf{x}(t),t)$ and $\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|$ converge exponentially fast to the origin.

It is desirable to notice that one can derive more precise conditions for exponential convergence of the estimates with algorithms (7) from the proofs of Theorems 1, 2. In particular in the proofs we neglected term $\beta(\mathbf{x},t)\mathcal{F}(\mathbf{x},\boldsymbol{\theta},\hat{\boldsymbol{\theta}})$ in the equations for derivatives $\hat{\boldsymbol{\theta}}$. The complete set of conditions would be, therefore,

as follows:

$$\exists L > 0, \ \delta > 0: \qquad \int_{t}^{t+L} (\mathcal{F}_{0}^{T}(\mathbf{x}(t), \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}(\tau), \tau) \mathcal{F}_{0}(\mathbf{x}(t), \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}(\tau), \tau) +$$

$$\beta(\mathbf{x}, t) \mathcal{F}(\mathbf{x}, \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}(\tau))) d\tau \ge \delta I \ \forall t > 0,$$
(10)

where matrix function $\mathcal{F}(\mathbf{x}, \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}(t))$ is defined as in Assumption 3 and function $\mathcal{F}_0(\mathbf{x}, \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}, t)$ is:

$$\mathcal{F}_0(\mathbf{x}, \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}, t) = \int_0^1 \frac{\partial f(\mathbf{x}, \mathbf{s}(\lambda), t)}{\partial \mathbf{s}} d\lambda, \quad \mathbf{s}(\lambda) = \boldsymbol{\theta}\lambda + \hat{\boldsymbol{\theta}}(1 - \lambda)$$

So far we have shown that, for the class of nonlinearly parameterized systems, there exist a control function and parameter adjustment algorithms such that solutions of the whole system are bounded, and parametric uncertainty is decreasing in time. We have shown also that in case of persistently excited functions $\alpha(\mathbf{x},t)$ the estimates $\hat{\theta}(t)$ in (7) converge exponentially fast to vector $\boldsymbol{\theta}$. These results will now be extended to a broader class of nonlinearities. We replace Assumptions 2, 4 with their locally verified versions.

Assumption 5 For the given nonlinear function $f(\mathbf{x}, \boldsymbol{\theta})$ there exits the following partition of the state space:

$$\Omega_{\mathbf{x}} = \Omega_M(\mathbf{x}) \cup \Omega_A, \quad \Omega_M(\mathbf{x}) = \bigcup_j \Omega_{M,j}(\mathbf{x}), \quad \Omega_A = \Omega_{\mathbf{x}}/\Omega_M(\mathbf{x})$$

where $\Omega_{M,j}(\mathbf{x}) = {\mathbf{x} | (\mathbf{x} - \mathbf{c}_j)^T (\mathbf{x} - \mathbf{c}_j) \le r_j^2}$ are the balls in \mathbb{R}^n where Assumptions 2, 4 are satisfied for every $\boldsymbol{\theta} \in \Omega_{\theta}$ and corresponding functions $\boldsymbol{\alpha}_j(\mathbf{x},t)$ and constants D_j , $D_{1,j}$.

A typical example of a nonlinear function which satisfies this assumption is $\sin(\theta x)$, where the unknown parameter θ belongs to a bounded interval. Another example is x^{θ} , $\theta \in [0, \infty)$. The last parametrization is widely used in modelling physical "power low" phenomena in nature (see, for example [30], where this function models effects of nonlinear damping in muscles).

In the sequel we will denote control functions (4) associated with parameter adjustment algorithms in $\Omega_{M,j}$ by symbol $u_{0,j}(\mathbf{x},t)$. Once Assumptions 2, 4 hold only locally, we can guarantee convergence of the estimates only if the state belongs to $\Omega_M(\mathbf{x})$. Therefore, extra control effort is needed. In order to specify the desired feedback acting in the domain Ω_A we introduce the following assumption on system (1) dynamics:

Assumption 6 For any $\mathbf{x}_0 \in \Omega_{\mathbf{x}}$ there exists a control function $u_j(\mathbf{x}, t)$ that steers the state \mathbf{x} of system (1) into the neighborhood of \mathbf{c}_j : $\|\mathbf{x} - \mathbf{c}_j\|^2 \le \delta_j^2$, $\delta < r_j^2$ in finite time.

It should be noticed, however, that this assumption does not require existence of stabilizing feedback, local or global in Lyapunov sense, at the points $\mathbf{x} = \mathbf{c}_i$.

Let us finally consider the following control/identification scheme:

$$\sigma_{j} = \begin{cases} 1 - \sigma_{j}, & \mathbf{x} = B_{j,\sigma_{j}} \\ \sigma_{j}, & \mathbf{x} \neq B_{j,\sigma_{j}} \end{cases}, \ \sigma_{j}(0) = \begin{cases} 0, & \|\mathbf{x}(0) - \mathbf{c}_{j}\| > \delta_{j} \\ 1, & \|\mathbf{x}(0) - \mathbf{c}_{j}\| \le \delta_{j} \end{cases}$$

$$B_{j,0} = \{\mathbf{x} : \|\mathbf{x} - \mathbf{c}_{j}\| = \delta_{j}\}, \ B_{j,1} = \{\mathbf{x} : \|\mathbf{x} - \mathbf{c}_{j}\| = r_{j}\}$$

$$u = (1 - \sigma_{j})u_{j}(\mathbf{x}, t) + \sigma_{j}u_{0,j}(\mathbf{x}, t)$$

$$\hat{\boldsymbol{\theta}} = \sigma_{j}\Gamma(\hat{\boldsymbol{\theta}}_{P}(\mathbf{x}, t) + \hat{\boldsymbol{\theta}}_{I}(t) + C_{j}(t));$$

$$\hat{\boldsymbol{\theta}}_{P}(\mathbf{x}, t) = \psi(\mathbf{x}, t)\boldsymbol{\alpha}_{j}(\mathbf{x}, t) - \Psi_{j}(\mathbf{x}, t)$$

$$\dot{\hat{\boldsymbol{\theta}}}_{I} = \sigma_{j}(\varphi(\psi(\mathbf{x}, t))\boldsymbol{\alpha}_{j}(\mathbf{x}, t) + \partial\Psi_{j}(\mathbf{x}, t)/\partial t - \psi(\mathbf{x}, t)(\partial\boldsymbol{\alpha}_{j}(\mathbf{x}, t)/\partial t) - (\psi(\mathbf{x}, t)L_{\mathbf{f}_{1}}\boldsymbol{\alpha}_{j}(\mathbf{x}, t) - L_{\mathbf{f}_{1}}\Psi_{j}(\mathbf{x}, t)) - (\psi(\mathbf{x}, t)L_{\mathbf{g}_{1}}\boldsymbol{\alpha}_{j}(\mathbf{x}, t) - L_{\mathbf{g}_{1}}\Psi_{j}(\mathbf{x}, t))u(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) + \beta_{j}(\mathbf{x}, t)(\mathbf{f}_{2}(\mathbf{x}, \hat{\boldsymbol{\theta}}) + \mathbf{g}_{2}(\mathbf{x})u(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)))$$

$$C_{j}(t) = (\theta_{P}(\mathbf{x}(t'_{j-1}), t'_{j-1}) - \theta_{P}(\mathbf{x}(t_{j}), t_{j}) + C_{j}(t'_{j-1})),$$
(11)

where t_i are the time instants when \mathbf{x} hits the domain $\|\mathbf{x} - \mathbf{c}_j\| = \delta_j$ for $\sigma_j = 1$ and $t_i' > t_i$ stands for the time moments when the state \mathbf{x} reaches $\|\mathbf{x} - \mathbf{c}_j\| = r_j$ (for $\sigma_j = 1$). Algorithm (11) includes algorithm (7) as a part. It also includes switching algorithm which specifies the time when parameter estimation procedure (7) shall be turned "on"/"off". The identifying properties of this new algorithm follow from Theorems 1, 2 and are formulated in the following corollary:

Corollary 1 Let Assumptions 1, 5 hold and there exist at least one $\alpha_j(\mathbf{x},t)$ such that Assumption 3 is satisfied. Then system (1), (11) trajectories are bounded. If, in addition, function $\alpha_j(\mathbf{x},t)$ is persistently exciting:

$$\exists \delta > 0: \int_{t}^{t+L} \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau) \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau)^{T} d\tau \ge \delta I \ \forall t > 0,$$
(12)

with sufficiently small L > 0, then parameters $\hat{\boldsymbol{\theta}}(t)$ converge to $\boldsymbol{\theta}$ $t \to \infty$ monotonically with respect to the norm $\|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\|$.

A consequence of Corollary 1 is that increase of excitation in functions $\alpha_j(\mathbf{x},t)$ results in an extension of the class of nonlinearities suitable for our approach. This is consistent with previously reported results [7] on parameter convergence in nonlinearly parameterized systems. Whether extension of the class of nonlinearities to more general functions renders it necessary to increase excitation, however, is still an open issue⁴.

For illustration consider the following system as an application of Corollary 1:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \sin(\theta x_1) + u, \tag{13}$$

⁴An example is constructed in [7], where nonlinear persistent excitation condition holds for the given parametrization, while the linear persistent excitation condition for linear parametrization with respect to the same parameter-independent function is not satisfied.

where parameter $\theta \in \Omega_{\theta} = [0.6, 1.4]$ is unknown a-priori. The goal is to design input $u(x_1, x_2, t)$ and estimator $\hat{\theta}(t)$ such that trajectories of the system are bounded and $\hat{\theta}(t) \to \theta$ as $t \to \infty$. For the given bounds of Ω_{θ} we first find the domain Ω_M , satisfying Assumption 5:

$$\Omega_M(\mathbf{x}) = \{\mathbf{x} \mid x_1 \in [-3.38, -2.59]\} \cup \{\mathbf{x} \mid x_1 \in [-1.14, -1.14]\} \cup \{\mathbf{x} \mid x_1 \in [2.59, 3.38]\} = \Omega_{M,1} \cup \Omega_{M,2} \cup \Omega_{M,3} \cup \Omega_{M,3} \cup \Omega_{M,4} \cup \Omega_{M,4}$$

Let us suppose that initial conditions of system (13) are located most closely to the subset $\Omega_{M,1}(\mathbf{x}) = \{\mathbf{x} \mid x_1 \in [-3.38, -2.59]\}$. Then it is natural to assume that the desired position of the plant for the purpose of identification is in the center of $\Omega_{M,1}: x_1 = x_1^* = -2.985$. Function $\psi(\mathbf{x}, t)$ satisfying Assumption 1 is chosen in the following manner: $\psi(x_1, x_2) = x_1 + x_2 - x_1^*$. Hence, according to (4), control inputs $u_{0,1}(\mathbf{x}, t)$, $u_1(\mathbf{x}, t)$ are given by equations:

$$u_{0,1}(\mathbf{x}) = -x_2 - \sin(\hat{\theta}, x_1) - \psi(x_1, x_2), \quad u_1(\mathbf{x}) = -x_2 - \psi(x_1, x_2) - \operatorname{sign}(\psi(x_1, x_2))$$

Function $\alpha(x_1, x_2) = -x_1$, and function $\hat{\theta}_P$, in (11) is as follows:

$$\hat{\theta}_P(\mathbf{x}) = \psi(x_1, x_2)\alpha(x_1, x_2) - \Psi(x_1, x_2), \ \Psi(x_1, x_2) = (x_1 - x_1^*)x_2 + \frac{x_2^2}{2}.$$

Function $\hat{\theta}_I$ now follows explicitly from equation (11).

In the next section we illustrate the application to and main steps in the design of our algorithms for two diverse, challenging and practically relevant problems. In the first example we apply our approach to the optimal slip identification problem in brake control systems. The second example provides a system for adaptive content-dependent filtering and classification of visual information.

4 Examples

Example 1. Braking wheel control problem. Consider the problem of minimizing the braking distance for a single wheel rolling along a surface. The surface properties can vary depending on the current position of the wheel. The wheel dynamics can be given by the following system of differential equations [24]:

$$\dot{x}_{1} = -\frac{1}{m}F_{s}(F_{n}, \mathbf{x}, \theta)
\dot{x}_{2} = \frac{1}{J}(F_{s}(F_{n}, \mathbf{x}, \theta)r - u)
\dot{x}_{3} = -\frac{1}{x_{1}}((\frac{1}{m}(1 - x_{3}) + \frac{r^{2}}{J})F_{s}(\mathbf{x}, \theta) - \frac{r}{J}u),$$
(14)

 x_1 is longitudinal velocity, x_2 is angular velocity, $x_3 = (x_1 - x_2)/x_1$ is wheel slip, m is mass of the wheel, J is moment of inertia, r is radius of the wheel, u is control input (brake torque), $F_s(F_n, \mathbf{x}, \theta)$ is a function specifying the tire-road friction force depending on the surface-dependent parameter θ and the load force F_n . This function, for example, can be derived from steady-state behavior of the LuGre tire-road friction model [6]:

$$F_s(F_n, \mathbf{x}, \theta) = F_n \operatorname{sign}(x_2) \frac{\frac{\sigma_0}{L} g(x_2, x_3, \theta) \frac{x_3}{1 - x_3}}{\frac{\sigma_0}{L} \frac{x_3}{1 - x_3} + g(x_2, x_3, \theta)},$$

$$g(x_2, x_3, \theta) = \theta(\mu_C + (\mu_S - \mu_C)e^{-\frac{|rx_2x_3|}{|1-x_3|v_S}}),$$

where μ_C , μ_S are Coulomb and static friction coefficients, v_s is the Stribeck velocity, σ_0 is the normalized rubber longitudinal stiffness, L is the length of the road contact patch. In order to avoid singularities we assume, as suggested in [24], that the system is turned off when velocity x_1 reaches a small neighborhood of zero (in our example we stopped simulations as soon as x_1 becomes less than 5 m/sec).

While the majority of the model parameters can be estimated a-priori, the tire-road parameter θ is dependent on the properties of the road surface. Therefore, on-line identification of the parameter θ is desirable in order to compute the optimal slip value

$$x_3^* = \arg\max_{x_3} F_s(F_n, \mathbf{x}, \theta) \tag{15}$$

which ensures the maximum deceleration force and therefore results in the shortest braking distance.

The main loop controller is derived in accordance with the standard certainty-equivalence principle and can be written as follows:

$$u(\mathbf{x}, \hat{\theta}, x_3^*) = \frac{J}{r} \left(\left(\frac{1}{m} (1 - x_3) + \frac{r^2}{J} \right) F_s(F_n, \mathbf{x}, \hat{\theta}) - K_s x_1 (x_3 - x_3^*) \right), K_s > 0$$

In order to estimate parameter θ by measuring the values of variables x_1, x_2 and x_3 , we construct the following subsystem:

$$\dot{\hat{x}}_3 = -\frac{1}{x_1} \left(\left(\frac{1}{m} (1 - x_3) + \frac{r^2}{J} \right) F_s(F_n, \mathbf{x}, \hat{\theta}) - \frac{r}{J} u \right) + (x_3 - \hat{x}_3)$$

and consider dynamics of the error function $\psi(\mathbf{x},t) = \psi(x_3,\hat{x}_3) = x_3 - \hat{x}_3$:

$$\dot{\psi} = -\psi + \frac{1}{r_1} \left(\left(\frac{1}{m} (1 - x_3) + \frac{r^2}{I} \right) (F_s(F_n, \mathbf{x}, \theta) - F_s(F_n, \mathbf{x}, \hat{\theta})) \right)$$
(16)

Function $\frac{1}{x_1}((\frac{1}{m}(1-x_3)+\frac{r^2}{J})F_s(\mathbf{x},\theta))$ is monotonic in θ and satisfies Assumptions 2, 4 with

$$\alpha(\mathbf{x},t) = \frac{1}{x_1} \left(\left(\frac{1}{m} (1 - x_3) + \frac{r^2}{J} \right) g(x_2, x_3, 1) \right)$$

Therefore, in order to design an estimation scheme satisfying assumptions of Theorem 2 we shall find functions $\Psi(\mathbf{x},t)$, $\beta(\mathbf{x},t)$ such that Assumption 3 holds. Notice that every equation in (14) depends on unknown parameter θ explicitly. Therefore, according to the introduced terminology, there is no uncertainty independent partition of system (14), i.e. $\mathbf{x} = \mathbf{x}_2$. Let us choose $\beta(\mathbf{x},t) = 0$. Then Assumption 3 reduces to the following equation:

$$\frac{\partial \Psi(\mathbf{x}, t)}{\partial \mathbf{x}} = \psi(\mathbf{x}, t) \frac{\partial \alpha(\mathbf{x})}{\partial \mathbf{x}}$$
(17)

Instead of trying to solve this equation explicitly for function $\Psi(\mathbf{x},t)$ we *embed* system (14), (16), as suggested in [26], into one of higher order, such that for the new set of equations Assumption 3 will be reduced to a case where $\alpha(\mathbf{x},t)$ does not depend explicitly on \mathbf{x}_2 . In fact, the problem with Assumption 3 would be solved

if we replace $\alpha(\mathbf{x},t)$ with the function of time $\xi(t)$: $\mathbb{R}_+ \to \mathbb{R}$, of which the derivative is known. Let us derive the required function $\xi(t)$. Notice that function $\alpha(\mathbf{x},t)$ is continuous in its arguments and, moreover, differentiable for $x_1 \geq 5$. Therefore, given that the right-hand side of system (14) is locally bounded, we can conclude that function $\alpha(\mathbf{x},t)$ has a bounded derivative for bounded \mathbf{x} . The state, moreover, is bounded as longitudinal velocity x_1 and angular velocity x_2 are bounded during the braking/acceleration regime, and relative slip x_3 is bounded by the way it is defined in (14). Therefore, it is possible to track signal $\alpha(\mathbf{x}(t),t)$ with arbitrary high precision by use of smooth high-gain estimators. If estimators with discontinuous right-hand sides are allowed then it is possible to provide exact tracking of $\alpha(\mathbf{x},t)$. Let us consider the following candidate for the estimator of $\alpha(\mathbf{x},t)$:

$$\dot{\xi} = -\frac{\partial \alpha}{\partial x_2} \frac{1}{J} u + \frac{\partial \alpha}{\partial x_3} \frac{1}{x_1} \frac{r}{J} u - K_{\xi} \varphi_{\xi} (\xi - \alpha(\mathbf{x}, t)), \tag{18}$$

where $K_{\xi} > 0$ and $\varphi_{\xi}(\xi - \alpha(\mathbf{x}, t))(\xi - \alpha(\mathbf{x}, t)) \ge 0$ are to be chosen to dominate the following sum

$$\left(-\frac{\partial \alpha}{\partial x_1}\frac{1}{m} + \frac{\partial \alpha}{\partial x_2}\frac{r}{J} - \frac{\partial \alpha}{\partial x_3}\frac{1}{x_1}\left(\left(\frac{1}{m}(1-x_3) + \frac{r^2}{J}\right)\right)F_s(F_n, \mathbf{x}, \theta)\right)$$

for $x_1 \in [5, 40]$, $x_2 \in [100, 1]$, $x_3 \in (0, 1)$, $\theta \in (0, 2]$ and $|\xi - \alpha(\mathbf{x}, t)| > \varepsilon_0 = 0.001$. Let us assume for the moment that function $\alpha(\mathbf{x}, t) = \xi(t)$. Taking this property into account we can extend system (16) with equation (18) and replace function $\alpha(\mathbf{x}, t)$ in description (7) of the algorithm with function $\xi(t)$ of which the derivative is known. Hence, for the new system Assumption 3 will be automatically satisfied. Then according to (7) and (16) parameter adjustment algorithm will be given by the following system:

$$\hat{\theta} = -\gamma((x_3 - \hat{x}_3)\xi + \hat{\theta}_I), \quad \gamma = 100$$

$$\dot{\hat{\theta}}_I = (x_3 - \hat{x}_3)(\xi - \dot{\xi})$$
(19)

The only difference between algorithm (19) and those which follow from explicit analytical solution of (17) is in the residual term $\varepsilon(t) = \xi - \alpha(\mathbf{x}, t)$, which can be made arbitrary small. On the other hand, according to Theorem 2 the original system (without replacing $\alpha(\mathbf{x}, t)$ with $\xi(t)$) is exponentially stable, which in turn guarantees convergence of the estimates $\hat{\theta}$ to the actual values of θ with algorithm (19) if $\varepsilon(t)$ is sufficiently small⁵.

We simulated system (14) – (19) with the following setup of parameters and initial conditions: $\sigma_0 = 200$, L = 0.25, $\mu_C = 0.5$, $\mu_S = 0.9$, $v_s = 12.5$, r = 0.3, m = 200, J = 0.23, $F_n = 3000$, $K_s = 30$. The effectiveness of estimation algorithm (19) could be illustrated with Figure 1. Estimates $\hat{\theta}$ approach the actual values of parameter θ sufficiently fast for the controller to calculate the optimal slip value x_3^* and steer the system toward this point in real braking time. Effectiveness of the proposed identification-based control can be confirmed by comparing the braking distance in the system with on-line estimation of x_3^* according to

⁵In this particular example in addition to the exponential stability argument one can easily derive from differential equations for $\hat{\theta}$: $\dot{\hat{\theta}} = -\gamma(\dot{\psi} + \psi)(\alpha(\mathbf{x}, t) + \varepsilon(t))$ from the proofs of Theorems 1, 2 that. This equation implies exponential convergence of $\hat{\theta}$ to θ , provided that $\alpha(\mathbf{x}, t) - \varepsilon(t) > \delta > 0$ for some positive constant δ

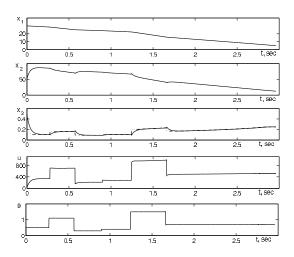


Figure 1: Plots of the trajectories of system (14)

formula (15) with $\theta = \hat{\theta}$ with the one, in which the values of x_3^* were kept constant (in the interval [0.1, 0.2]). For model parameters as presently given and road condition given by the piece-wise constant function

$$\theta(s) = \begin{cases} 0.3, & s \in [0, 8] \\ 1.3, & s \in (8, 16] \\ 0.7, & s \in (16, 24] \\ 0.4, & s \in (24, 32] \\ 1.5, & s \in (32, 40] \\ 0.6, & s \in (40, \infty] \end{cases}, s = \int_0^t x_1(\tau) d\tau$$

the simulated braking distance obtained with our on-line estimation procedure of x_3^* is 54.95 meters. This result compares favorably with the values obtained for preset values of x_3^* , which range between 57.52 and 55.32 (for $x_3^* = 0.1$ and $x_3^* = 0.2$ respectively).

Example 2. Classification of occluded and linearly nonseparable patterns. Another illustrative application of our parameter identification scheme is taken from the field of neural computation and control of biological systems. In these domains functions that are nonlinear in their parameters are widely used. We will discuss an example involving a model of visual object recognition system with adaptive identification of local spatial features of the presented objects.

The problem is to identify two visual patterns given that they may be out of focus (blurred), partially occluded by each other, slightly distorted, or incomplete. A typical classification system for this purpose consists of a two-layer neural network in which the first, sensory layer feeds its outputs to a layer of decision units.

We modeled a simple pattern recognition system, consisting of a pattern input system and two image template systems. In the input system, visual information arrives initially in a two-dimensional array (i, j) of sensors. The output of each sensor is mapped onto the first layer, which consists of a two-dimensional array (i, j) of filters. The connectivity of the sensors to the filters is one-to-all, but assures topographical projection by means of connection weights. Maximum weight is given to topographically corresponding units,

neighboring ones receive exponentially declining weights. The process is functionally equivalent to spatial integration with an exponential kernel. The output of these filters is projected topographically (one-to-one) onto the second layer, which consists of a two-dimensional array of decision units. Connections within each layer have not been modeled at any of these levels. The architecture of the template systems is identical to that of the input system, except that each of the sensors is replaced by a binary value ("off" or "on"), corresponding to the template of an image stored in memory. These images can be represented by binary matrices P_1 and P_2 correspondingly.

The input and template systems are connected at the level of the first and second layer. Connections at the level of the filters layer are one-to-one, reciprocal but not symmetrical, between the filters and their counterparts within each of the template systems. Connections at the level of the decision layer are all-to-all, reciprocal and symmetrical (for simplicity) between the decision units and their counterparts in the template systems. Also the two template systems are connected to each other in this manner at this level.

Whereas the sensory units locally filter the spatial information in the input, the decision nodes match it with the templates. The information in the input system matches either of the templates, to the degree that its units are synchronized with those of either template. In addition, the location of the synchronized nodes in the decision units indicates where the matching occurs in the input.

The decision units and their counterparts in the template systems can be modelled by the following ensemble of Hindmarsh and Rose spiking neurons [14]⁶:

$$\dot{x}_{1,k} = -ax_{1,k}^3 + bx_{1,k}^2 + x_{4,k} + x_{2,k} - x_{3,k} + u_k + I_0$$

$$\dot{x}_{2,k} = c - dx_{1,k}^2 - x_{2,k}$$

$$\dot{x}_{3,k} = \varepsilon(s(x_{1,k} + x_0) - x_{3,k}),$$

$$u_k = \gamma(x_{1,m} + x_{1,r} - 2x_{1,k}), \quad m \neq r, k \neq m, \quad k, m, r \in \{1, 2, 3\}$$
(20)

Parameters $a, b, c, d, s, x_0, \epsilon, I_0$ are all positive constants with the following values: $a = 1, b = 3, c = 1, d = 5, s = 4, x_0 = 1.6, \varepsilon = 0.001, I_0 = 1.4$ as specified by [8]. Function $u_k = -2x_{1,k} + \hat{x}_{1,k} + \bar{x}_{1,k}$ is the coupling function, variables $\hat{x}_{1,k}$, $\bar{x}_{1,k}$ are the outputs of the template systems at the level of the decision units and their corresponding coupling functions are $\hat{u}_k = -2\hat{x}_{1,k} + x_{1,k} + \bar{x}_{1,k}, \bar{u}_k = -2\bar{x}_{1,k} + \hat{x}_{1,k} + x_{1,k}$.

Variable $x_{4,k}$ stands for input dependent current produced by the sensory cell:

$$\dot{x}_{4,k} = \frac{1}{\tau} (\beta - x_{4,k} + r(\theta_0, \mathbf{s}_k(t)))
r(\theta_0, \mathbf{s}_k(t)) = \sum_{i=1}^N \sum_{j=1}^N e^{-\frac{|i(k)-i|+|j(k)-j|}{\theta_0}} s_{k,i,j} \delta(t - \tau_{i,j}), \ \beta > 0$$
(21)

i(k), j(k) specify the position of the k-th unit, $\tau > 0$ is the integration parameter, $s_{k,i,j}$ denotes intensity of the (i, j)-th element in the image, $\delta(t - \tau_{i,j}) : \mathbb{R} \to \mathbb{R}_{\geq 0}$ stand for the pattern-induced signals (impulses

⁶ For the sake of compactness in the text hereafter we omit indices i, j in the subscripts of the system variables

of unit amplitude and width $\Delta = 0.05T$ at $t = \tau_{i,j}$, where T is the period of generation of each impulse). Exponential functions in $r(\theta_0, \mathbf{s}_k(t))$ represent the distribution of the weights. Numbers $\tau_{i,j}$ stand for time-delays in the transmission of the signal from a sensor to the filters. This delay is variable due to the difference in properties of the transmission cables. It is given by the ration of cable length and width, which is a simplification of actual signal transmission on neural systems. These delays together with the exponentially decaying amplitudes of $s_{k,i,j}$ in space form the receptive field of a filter. One of the main properties of such an organization is that input signal $\mathbf{s}_k(t)$ is distributed in time and space, providing in principle a unique spatiotemporal signature for every different static visual pattern.

For the template systems the pattern-induced currents evolve according to the following equations

$$\dot{\hat{x}}_{4,k} = \frac{1}{\tau} \left(-\beta \hat{x}_{4,k} + r(\theta_0, \hat{\mathbf{s}}_k(t, \theta_1)) \right),
\dot{\bar{x}}_{4,k} = \frac{1}{\tau} \left(-\beta \bar{x}_{4,k} + r(\theta_0, \bar{\mathbf{s}}_k(t, \theta_1)) \right) \beta > 0$$

where functions $\hat{\mathbf{s}}_k(t)$ and $\bar{\mathbf{s}}_k(t)$ are the outputs of the spacial filters with parameter θ_1 :

$$\hat{s}_{k,i,j}(t) = \delta(t - \tau_{i,j}) \sum_{m=1}^{N} \sum_{r=1}^{N} e^{\frac{-|i-m|-|r-j|}{\theta_1}} P_{1,m,r}, \quad \bar{s}_{k,i,j}(t) = \delta(t - \tau_{i,j}) \sum_{m=1}^{N} \sum_{r=1}^{N} e^{\frac{-|i-m|-|r-j|}{\theta_1}} P_{2,m,r}$$

This filters model effects of changes in intensity of the light, focal adaptation and sharpness of the templates in presented visual patterns.

The problem for such architecture is the following: if the picture is not stable in time and perturbed by unmeasured changes in focus, then how can the decision units reach detectable synchrony?

Technically, the solution would be to adjust parameters θ_1 in (22) in response to distortion in the input patters. The difficulty, however, is that the functions $r(\theta, \hat{\mathbf{s}}_k(t, \theta_1))$ $r(\theta, \bar{\mathbf{s}}_k(t, \theta_1))$ are nonlinear in parameter θ_1 . Classical linear identification schemes result in a prohibitively large dimension of the estimator (in our simplified example a 100×100 sensory field will require 10^{10} independent parameters in each cell). A further problem is that of performance in terms of robust identification of the parameter for slightly distorted patterns, if no persistent excitation in $\mathbf{s}_k(t)$ is assumed. For these reasons we would require new adjustment algorithms to estimate parameters $\hat{\theta}_1$, $\bar{\theta}_1$ in the junctions of nodes (20), (21) in order to ensure adaptive properties of the classifier together with practical realizability and reliability.

In order to derive the estimation algorithms for the parameters θ_1 in the templates we introduce the following error functions: $\hat{\psi}_k = x_{4,k} - \hat{x}_{4,k}$, $\bar{\psi}_k = x_{4,k} - \bar{x}_{4,k}$. Dynamics of function $\hat{\psi}_k(t)$, for example, follows from (22) and (21):

$$\dot{\hat{\psi}} = -\frac{\beta}{\tau}\hat{\psi} + (r(\theta_0, \mathbf{s}_k(t)) - r(\theta_0, \hat{\mathbf{s}}_k(t, \hat{\theta}_1)))$$
(22)

If the image contains the distorted template locally then the following equality holds: $\mathbf{s}_k(t) = \hat{\mathbf{s}}_k(t, \hat{\theta}_1)$, and equation (22) becomes as follows:

$$\dot{\hat{\psi}} = -\frac{\beta}{\tau}\hat{\psi} + (r(\theta_0, \hat{\mathbf{s}}_k(t, \theta_1)) - r(\theta_0, \hat{\mathbf{s}}_k(t, \hat{\theta}_1)))$$
(23)

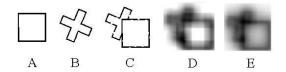


Figure 2: Reference patterns of the template system (A,B); distorted, combined pattern (C); blurred versions of the distorted pattern (D,E)

Function $\hat{\mathbf{s}}_k(t, \hat{\theta}_1)$ is monotonic in θ_1 with respect to θ_1 (i.e. with constant $\alpha(\mathbf{x}, t)$). Hence, Assumptions 2, 4 are at least locally satisfied. Moreover, function $\alpha(\cdot)$ does not depend on \mathbf{x} , so Assumption 3 is satisfied as well. Therefore, applying Theorem 2 we can derive parameter adjustment algorithm in the following form

$$\hat{\theta}_{1} = x_{4,k} - \hat{x}_{4,k} + \hat{\theta}_{I,1}
\dot{\hat{\theta}}_{I,1} = \frac{\beta}{\tau} (x_{4,k} - \hat{x}_{4,k})$$
(24)

According to Theorem 2, algorithm (24) combined with (22) result in the desired estimator of θ_1 , which in addition guarantees exponential stability of the whole system with respect to the small perturbations in the presented patterns.

To illustrate the performance of our classifier, a square and a cross (Figures 2 A and B) were used as reference patterns in the template system. They were distorted and combined, one partially occluding the other as in Figure 2 C. Blurred versions of Figure 2 C, shown in 2 D and E, were presented to the system. The task was to recognize the patterns at their corresponding locations. In our simulations we used the following values of the model parameters: $\beta = 0.02$, $\tau = 0.01$, $\tau_{i,j}$ were set chosen in the interval from 0 to 100, $\gamma = 1$, $x_{1,k}(0) = -1.6$, $x_{2,k}(0) = -11.83$, $x_{3,k}(0) = 1.46$, $I_0 = 1.4$, $x_{4,k}(0) = 0$, $\hat{\theta}_I(0) = 1$.

Figure 3 shows the responses of two of the decision nodes. Those decision nodes that are topographical projections from regions where the square appeared are synchronized with their counterparts in the "square" template system. Likewise, those which correspond to regions where the cross appeared were synchronized with their counterparts in the "cross" template system. The synchrony occurs because the for the counterparts the parameters $\hat{\theta}_1$ (or $\bar{\theta}_1$) converge to their true values. Evolution of the estimates of θ_1 is shown in Figure 4.

Decision nodes at regions where no pattern was presented and their counterparts in the template systems fail to reach synchrony. The **theta parameters of these units, however, remain in a bounded domain.

5 Conclusion

We proposed a new class of parameterizations for nonlinearly parameterized models. Instead of aiming at a general solution for the problem of nonlinearity in the parameters, parametrization was restricted to a set of smooth functions, which are monotonic with respect to a linear functional in the parameters. For

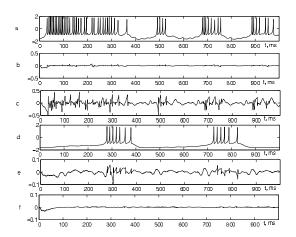


Figure 3: Response of the decision nodes located in points (90, 90) (plots a–c) and (5, 20) (plots d–e) respectively. Plots a, d contain trajectories $x_{1,k}(t)$ of the decision cells corresponding to the actual image, plots b, e reflect differences in the responses between the actual scene and memorized pattern "rectangle", plots c, f show deviations in perception of the scene and pattern "cross"

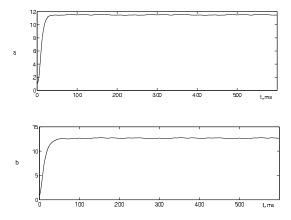


Figure 4: Trajectories of $\hat{\theta}_1(t)$ in the decision cell located at the point (90,90) (plot a), and $\bar{\theta}_1(t)$ in the decision cell located at the point (5,20) (plot b).

this new class, estimation algorithms were introduced and analyzed. It was been shown that standard linear persistent excitation conditions suffice to ensure exponentially fast convergence of the estimates to the actual values of unknown parameters. If, however, the monotonicity assumption holds only locally in the system state space, excitation with sufficiently high-frequency of oscillations is needed to ensure convergence. It is also desirable to notice that in case of linear parametrization the proposed parameter estimation schemes allow to estimate the unknowns in a dynamical system without asking for usual filtered transformations, thus reducing the number of integrators in the estimator.

Two rather distinct applications of our method were provided as examples. One is devoted to on-line identification of the optimal slip in a braking wheel. The second example touches on the problem of dynamic recognition of visual patterns in artificial neural networks. Both problems may be considered to have practical significance. The effectiveness of the solution to these problems leads us to expect that this method can successfully be implemented in other applications.

6 Appendix

Proof of Theorem 1. Let us first calculate time-derivative of function $\hat{\boldsymbol{\theta}}(\mathbf{x},t)$: $\dot{\hat{\boldsymbol{\theta}}}(\mathbf{x},t) = \Gamma(\dot{\hat{\boldsymbol{\theta}}}_P + \dot{\hat{\boldsymbol{\theta}}}_I) = \Gamma(\dot{\boldsymbol{\psi}}\boldsymbol{\alpha}(\mathbf{x},t) + \boldsymbol{\psi}\dot{\boldsymbol{\alpha}}(\mathbf{x},t) - \dot{\boldsymbol{\Psi}}(\mathbf{x},t) + \dot{\hat{\boldsymbol{\theta}}}_I)$. Notice that

$$\psi \dot{\boldsymbol{\alpha}}(\mathbf{x},t) - \dot{\boldsymbol{\Psi}}(\mathbf{x},t) + \dot{\hat{\boldsymbol{\theta}}}_{I} = \psi(\mathbf{x},t) \frac{\partial \boldsymbol{\alpha}(\mathbf{x},t)}{\partial \mathbf{x}_{1}} \dot{\mathbf{x}}_{1} + \psi(\mathbf{x},t) \frac{\partial \boldsymbol{\alpha}(\mathbf{x})}{\partial \mathbf{x}_{2}} \dot{\mathbf{x}}_{2} + \psi(\mathbf{x},t) \frac{\partial \boldsymbol{\alpha}(\mathbf{x},t)}{\partial t} - \frac{\partial \boldsymbol{\Psi}(\mathbf{x},t)}{\partial \mathbf{x}_{1}} \dot{\mathbf{x}}_{1} - \frac{\partial \boldsymbol{\Psi}(\mathbf{x},t)}{\partial \mathbf{x}_{2}} \dot{\mathbf{x}}_{2} - \frac{\partial \boldsymbol{\Psi}(\mathbf{x},t)}{\partial t} + \dot{\hat{\boldsymbol{\theta}}}_{I}$$
(25)

According to Assumption 3, $\frac{\partial \Psi(\mathbf{x},t)}{\partial \mathbf{x}_2} = \psi(\mathbf{x},t) \frac{\partial \alpha(\mathbf{x},t)}{\partial \mathbf{x}_2} + \beta(\mathbf{x},t)$. Then taking into account (25), we can obtain

$$\psi \dot{\boldsymbol{\alpha}}(\mathbf{x},t) - \dot{\boldsymbol{\Psi}}(\mathbf{x},t) + \dot{\hat{\boldsymbol{\theta}}}_{I} = \left(\psi(\mathbf{x},t)\frac{\partial \boldsymbol{\alpha}(\mathbf{x},t)}{\partial \mathbf{x}_{1}} - \frac{\partial \boldsymbol{\Psi}}{\partial \mathbf{x}_{1}}\right) \dot{\mathbf{x}}_{1} + \psi(\mathbf{x},t)\frac{\partial \boldsymbol{\alpha}(\mathbf{x},t)}{\partial t} - \frac{\boldsymbol{\Psi}(\mathbf{x},t)}{\partial t} - \beta(\mathbf{x},t)(\mathbf{f}_{2}(\mathbf{x},\boldsymbol{\theta}) + \mathbf{g}_{2}(\mathbf{x})u) + \dot{\hat{\boldsymbol{\theta}}}_{I}$$
(26)

Notice that according to the proposed notation we can rewrite the term $\left(\psi(\mathbf{x},t)\frac{\partial \boldsymbol{\alpha}(\mathbf{x},t)}{\partial \mathbf{x}_1} - \frac{\partial \Psi}{\partial \mathbf{x}_1}\right)\dot{\mathbf{x}}_1$ in the following form: $(\psi(\mathbf{x},t)L_{\mathbf{f}_1}\boldsymbol{\alpha}(\mathbf{x},t) - L_{\mathbf{f}_1}\Psi(\mathbf{x},t)) + (\psi(\mathbf{x},t)L_{\mathbf{g}_1}\boldsymbol{\alpha}(\mathbf{x},t) - L_{\mathbf{g}_1}\Psi(\mathbf{x},t))u(\mathbf{x},\hat{\boldsymbol{\theta}},t)$. Hence it follows from (7) and (26) that $\psi\dot{\boldsymbol{\alpha}}(\mathbf{x},t) - \dot{\Psi}(\mathbf{x},t) + \dot{\boldsymbol{\theta}}_I = \varphi(\psi)\boldsymbol{\alpha}(\mathbf{x},t) - \beta(\mathbf{x},t)(\mathbf{f}_2(\mathbf{x},\boldsymbol{\theta}) - \mathbf{f}_2(\mathbf{x},\hat{\boldsymbol{\theta}}))$. Therefore derivative $\dot{\boldsymbol{\theta}}(\mathbf{x},t)$ can be written in the following way:

$$\dot{\hat{\boldsymbol{\theta}}} = \Gamma((\dot{\psi} + \varphi(\psi))\boldsymbol{\alpha}(\mathbf{x}, t) - \beta(\mathbf{x}, t)(\mathbf{f}_2(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{f}_2(\mathbf{x}, \hat{\boldsymbol{\theta}})))$$
(27)

Consider the following positive-definite function: $V_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \frac{1}{2} ||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}||_{\Gamma^{-1}}^2$. Its time-derivative according to equations (27) can be obtained as follows:

$$\dot{V}_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = (\varphi(\psi) + \dot{\psi})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \boldsymbol{\alpha}(\mathbf{x}, t) - (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \boldsymbol{\beta}(\mathbf{x}, t) (\mathbf{f}_2(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{f}_2(\mathbf{x}, \hat{\boldsymbol{\theta}}))$$

Let $\beta(\mathbf{x},t) \neq 0$, then consider the following difference $\mathbf{f}_2(\mathbf{x},\boldsymbol{\theta}) - \mathbf{f}_2(\mathbf{x},\hat{\boldsymbol{\theta}})$. Applying Hadamard's lemma we

represent this difference in the following way:

$$\mathbf{f}_2(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{f}_2(\mathbf{x}, \hat{\boldsymbol{\theta}}) = \int_0^1 \frac{\partial \mathbf{f}_2(\mathbf{x}, \mathbf{s}(\lambda))}{\partial \mathbf{s}} d\lambda (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}), \ \mathbf{s}(\lambda) = \boldsymbol{\theta}\lambda + \hat{\boldsymbol{\theta}}(1 - \lambda)$$

Therefore, according to Assumption 3 function $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \beta(\mathbf{x}, t) (\mathbf{f}_2(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{f}_2(\mathbf{x}, \hat{\boldsymbol{\theta}}))$ is positive semi-definite, hence using Assumption 2 and equality (3) we can estimate derivative $\dot{V}_{\hat{\boldsymbol{\theta}}}$ as follows

$$\dot{V}_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = -(f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t))(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{T} \boldsymbol{\alpha}(\mathbf{x}, t) \leq -D(f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t))^{2} = -D(\varphi(\psi) + \dot{\psi})^{2} \leq 0 (28)$$

Therefore $V_{\hat{\theta}}$ is non-increasing (property P2) is proven). Furthermore, integration of $\dot{V}_{\hat{\theta}}$ with respect to time results in

$$V_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}(0),\boldsymbol{\theta}) - V_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}(t),\boldsymbol{\theta}) \ge D \int_0^t (\dot{\psi}(\tau) + \varphi(\psi(\tau)))^2 d\tau \ge 0.$$

Function $V_{\hat{\theta}}$ is non-increasing and bounded from below as $V_{\hat{\theta}} \geq 0$, therefore

$$D\int_0^t (\dot{\psi}(\tau) + \varphi(\psi(\tau)))^2 d\tau \le V_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}(0), \boldsymbol{\theta}) < \infty.$$

Hence $(\varphi(\psi) + \dot{\psi}) = (f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)) = (f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)) \in L_2 \text{ (property P3)}).$

To prove property P1) let us consider the following function: $V(\psi, \hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = 2DQ(\psi) + V_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$, where $Q(\psi) = \int_0^{\psi} \varphi(\varsigma) d\varsigma$. Function $V(\psi, \hat{\boldsymbol{\theta}})$ is positive-definite with respect to $\psi(\mathbf{x}, t)$ and $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$. Its time-derivative obeys inequality: $\dot{V}(\psi, \hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) \leq 2D\varphi(\psi)\dot{\psi} - D(\dot{\psi} + \varphi(\psi))^2 = -D\varphi^2(\psi) - D\dot{\psi}^2 \leq 0$.

Therefore, function $V(\psi, \hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$ is bounded and non-increasing. Furthermore

$$\infty > V(\psi(\mathbf{x}(0), 0), \hat{\boldsymbol{\theta}}(0), \boldsymbol{\theta}) \ge V(\psi(\mathbf{x}(0), 0), \hat{\boldsymbol{\theta}}(0), \boldsymbol{\theta}) - V(\psi(\mathbf{x}(t), t), \hat{\boldsymbol{\theta}}(t), \boldsymbol{\theta}) \ge D \int_0^t \varphi^2(\psi(\mathbf{x}(\tau), \tau)) d\tau \ge 0$$

$$\infty > V(\psi(\mathbf{x}(0), 0), \hat{\boldsymbol{\theta}}(0), \boldsymbol{\theta}) \ge V(\psi(\mathbf{x}(0), 0), \hat{\boldsymbol{\theta}}(0), \boldsymbol{\theta}) - V(\psi(\mathbf{x}(t), t), \hat{\boldsymbol{\theta}}(t), \boldsymbol{\theta}) \ge D \int_0^t \dot{\psi}^2(\tau) d\tau \ge 0. \tag{29}$$

or, equivalently, $\dot{\psi}(t) \in L_2$, $\varphi(\psi(t)) \in L_2$. Hence, property P1) is proven as well. The L_2 norm bounds (8) for $\varphi(\psi)$ and $\dot{\psi}$ follow immediately from inequality (29):

$$\|\varphi(\psi)\|_2^2 \leq D^{-1}V(\psi(\mathbf{x}(0),0),\hat{\boldsymbol{\theta}}(0),\boldsymbol{\theta}), \ \|\dot{\psi}\|_2^2 \leq D^{-1}V(\psi(\mathbf{x}(0),0),\hat{\boldsymbol{\theta}}(0),\boldsymbol{\theta})$$

The L_{∞} norm bound for $\psi(\mathbf{x}(t),t)$ results from the inequality: $V(\psi(\mathbf{x}(0),0),\hat{\boldsymbol{\theta}}(0),\boldsymbol{\theta})-V(\psi(\mathbf{x}(t),t),\hat{\boldsymbol{\theta}}(t),\boldsymbol{\theta})\geq 0$. Consider function Λ defined as $\Lambda(d)=\max_{|\psi|}\{|\psi|\mid\int_{0}^{|\psi|}\varphi(\varsigma)d\varsigma=d\}$ and notice that it is monotonic and nondecreasing. Therefore, given that $\int_{0}^{\psi(\mathbf{x}(t),t)}\varphi(\varsigma)d\varsigma\leq\frac{1}{2D}V(\psi(\mathbf{x}(0),0),\hat{\boldsymbol{\theta}}(0),\boldsymbol{\theta})$ we can conclude that $|\psi|\leq \Lambda\left(\frac{1}{2D}V(\psi(\mathbf{x}(0),0),\hat{\boldsymbol{\theta}}(0),\boldsymbol{\theta})\right)$. To prove property P4) notice that function $V(\psi(\mathbf{x}(t),t),\hat{\boldsymbol{\theta}}(t),\boldsymbol{\theta})$ is bounded. Hence, as follows from condition (5), function $\psi(\mathbf{x}(t),t)$ is bounded as well. According to Assumption 1 boundedness of $\psi(\mathbf{x}(t),t)$ implies boundedness of the state \mathbf{x} . In addition it is assumed that $f(\mathbf{x},\hat{\boldsymbol{\theta}},t)$ is locally bounded with respect to $\mathbf{x},\hat{\boldsymbol{\theta}}$ and uniformly bounded in t. Therefore the difference $f(\mathbf{x},\boldsymbol{\theta},t)-f(\mathbf{x},\hat{\boldsymbol{\theta}},t)$ is bounded. Furthermore, according to (5), function $\varphi(\psi)\in C^0$ and therefore, given that ψ is bounded, this function is bounded as well. Hence $\dot{\psi}$ is bounded and by applying Barbalat's lemma one can show that $\psi(\mathbf{x}(t),t)\to 0$ at $t\to\infty$.

To compete the proof of the theorem (property P5) consider the difference $f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)$. Let function $\varphi \in C^1$, function $f(\mathbf{x}, \boldsymbol{\theta}, t)$ is differentiable in \mathbf{x} , $\boldsymbol{\theta}$; derivative $\partial f(\mathbf{x}, \boldsymbol{\theta}, t)/\partial t$ is bounded uniformly in t; function $\alpha(\mathbf{x}, t)$ is locally bounded with respect to \mathbf{x} and uniformly bounded with respect to t, then $d/dt(f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t))$ is bounded. On the other hand there exists the following limit

$$\lim_{t \to \infty} \int_0^t (f(\mathbf{x}, \boldsymbol{\theta}, \tau) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, \tau))^2 = \int_0^\infty (f(\mathbf{x}, \boldsymbol{\theta}, \tau) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, \tau))^2 \le \frac{1}{D} V_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}(0), \boldsymbol{\theta})$$

as $\int_0^t (f(\mathbf{x}, \boldsymbol{\theta}, \tau) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, \tau))^2$ is non-decreasing and bounded from above. Hence by Barbalat's lemma it follows that $f(\mathbf{x}, \boldsymbol{\theta}, \tau) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, \tau) \to 0$ as $t \to \infty$. Notice also that $\psi(\mathbf{x}(t), t) \to 0$ as $t \to \infty$. Then $\dot{\psi} \to 0$ as $t \to \infty$. The theorem is proven.

Proof of Theorem 2. Consider the following integral $\int_0^t (\dot{\psi}(\tau) + \varphi(\psi(\tau))^2 d\tau$. It was shown in Theorem 1 proof that $\int_0^t (\dot{\psi}(\tau) + \varphi(\psi(\tau))^2 d\tau \le \frac{1}{2D} ||\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}||_{\Gamma^{-1}}^2$ along system (1), (4), (27) solutions. Let us define $\mu(t) = \dot{\psi}(t) + \varphi(\psi(t))$, or

$$\dot{\psi} = -\varphi(\psi) + \mu(t),\tag{30}$$

where $\int_0^\infty \mu^2(\tau)d\tau \leq \frac{1}{2D}\|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|_{\Gamma^{-1}}^2$. According to the theorem conditions, $\varphi(\psi) = K\psi$, it is possible to derive the solution of equation (30) as follows $\psi(t) = \psi(0)e^{-Kt} + \int_0^t e^{-K(t-\tau)}\mu(\tau)d\tau$. Hence

$$|\psi(t)| \leq |\psi(0)|e^{-Kt} + \sqrt{\left(\int_0^t e^{-K(t-\tau)}\mu(\tau)d\tau\right)^2} \leq |\psi(0)|e^{-Kt} + \sqrt{\int_0^t e^{-2K(t-\tau)}d\tau} \int_0^t \mu^2(\tau)d\tau$$

$$\leq |\psi(0)|e^{-Kt} + \frac{1}{2}\sqrt{\frac{1}{KD}\|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|_{\Gamma^{-1}}^2}.$$
(31)

Property P6) is thus proven. In order to prove property P7) consider

$$\dot{\hat{\boldsymbol{\theta}}} = \Gamma(\dot{\psi} + \varphi(\psi))\boldsymbol{\alpha}(\mathbf{x}, t) = \Gamma(f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t))\boldsymbol{\alpha}(\mathbf{x}, t).$$

Function

$$D_1|\boldsymbol{\alpha}(\mathbf{x},t)^T(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})| \leq |f(\mathbf{x},\boldsymbol{\theta},t) - f(\mathbf{x},\hat{\boldsymbol{\theta}},t))| \leq D|\boldsymbol{\alpha}(\mathbf{x},t)^T(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})|$$
$$\boldsymbol{\alpha}(\mathbf{x},t)^T(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})(f(\mathbf{x},\hat{\boldsymbol{\theta}},t) - f(\mathbf{x},\boldsymbol{\theta},t)) > 0 \ \forall \ f(\mathbf{x},\boldsymbol{\theta},t) \neq f(\mathbf{x},\hat{\boldsymbol{\theta}},t).$$

Therefore, there exists $D_1 \leq \kappa(t) \leq D$ such that

$$\dot{\hat{\boldsymbol{\theta}}} = -\kappa(t)\Gamma\boldsymbol{\alpha}(\mathbf{x}, t)^T(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\boldsymbol{\alpha}(\mathbf{x}, t) = -\kappa(t)\Gamma\boldsymbol{\alpha}(\mathbf{x}, t)\boldsymbol{\alpha}(\mathbf{x}, t)^T(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}).$$

Hence

$$\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta} = e^{-\Gamma \int_0^t \kappa(\tau) \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau) \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau)^T d\tau} (\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta})$$
(32)

Consider the integral $\Gamma \int_0^t \kappa(\tau) \alpha(\mathbf{x}(\tau), \tau) \alpha(\mathbf{x}(\tau), \tau)^T d\tau$ for t > L

$$\Gamma \int_0^t \kappa(\tau) \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau) \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau)^T d\tau \ge \Gamma D_1 \int_0^t \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau) \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau)^T d\tau,$$

⁷We substitute the arguments of the functions $\dot{\psi}(\cdot)$ and $\psi(\cdot)$ with t. This means that we consider them as functions of time.

where $\alpha(\mathbf{x}(t),t)$ is persistently exciting. For any t>L there exists integer $n\geq 0$ such that t=nL+r, $r\in R, 0\leq r< L$. Therefore

$$\Gamma D_1 \int_0^t \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau) \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau)^T d\tau \ge \Gamma D_1 n \delta I \ge \left(\frac{\Gamma D_1 \delta}{L} t - I\right).$$

Then taking into account (32) one can write

$$\|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\| \le \|e^{\left(-\frac{\Gamma D_1 \delta}{L} t + I\right)}\|\|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|,\tag{33}$$

i. e. $\hat{\boldsymbol{\theta}}(t)$ converges to $\boldsymbol{\theta}$ exponentially fast. It means that there exist positive constants $\lambda > 0$, $\lambda \neq K$ and $D_{\hat{\boldsymbol{\theta}}} > 0$ such that $\|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\| \le e^{-\lambda t} \|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\| D_{\hat{\boldsymbol{\theta}}}$. It follows from Theorem 1 that $\psi(\mathbf{x}(t), t)$ is bounded. In addition due to Assumption 1 we can conclude that \mathbf{x} is bounded as well. By the theorem assumptions function $\boldsymbol{\alpha}(\mathbf{x}, t)$ is locally bounded with respect to \mathbf{x} and uniformly bounded in t. Therefore, there exists $D_{\boldsymbol{\alpha}} > 0$ such that $|\boldsymbol{\alpha}(\mathbf{x}, t)^T(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta})| \le D_{\boldsymbol{\alpha}} \|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\|$. Taking into account that $f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) = \mu(t)$ and $|f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)| \le D|\boldsymbol{\alpha}(\mathbf{x}, t)^T(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta})|$ we can derive from (30) the following estimate

$$|\psi(t)| \le |\psi(0)|e^{-Kt} + \|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|D_{\hat{\boldsymbol{\theta}}}D_{\alpha}D\int_{0}^{t} e^{-K(t-\tau)}e^{-\lambda\tau}d\tau \le |\psi(0)|e^{-Kt} + \frac{D_{\hat{\boldsymbol{\theta}}}D_{\alpha}D}{K-\lambda}\|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|e^{-\lambda t}$$
(34)

The theorem is proven.

Corollary 1 proof. In order to prove the corollary, we notice first that function σ_j is equal to unit for the following segments of the system solutions: $\|\mathbf{x}(\mathbf{x}_0, t_0, t) - \mathbf{c}_j\| < r_j$, $\|\mathbf{x}_0 - \mathbf{c}_j\| \le \delta_j$. Let us consider two cases: 1) $\|\mathbf{x}(\mathbf{x}_0, t_0, t) - \mathbf{c}_j\| < r_j$ for any $t > t_0$, and 2) for any t_0 and $\|\mathbf{x}(\mathbf{x}_0, t_0, t_0) - \mathbf{c}_j\| < r_j$ there exist $t_1 > t_0$ such that $\|\mathbf{x}(\mathbf{x}_0, t_0, t) - \mathbf{c}_j\| = r_j$.

In the first case Theorem 2 explicitly applies and the corollary follows automatically. In the second case, we can derive from Theorem 2 that $\psi(\mathbf{x},t)$ is bounded for every $t \in [t_0,t_1]$. Furthermore, according to the properties of function $\psi(\mathbf{x},t)$, it is bounded in t for every $\mathbf{x}: \|\mathbf{x} - \mathbf{c}_j\| \leq \delta_j$. Let us denote this bound by symbol Δ_{ψ} . Therefore, according to Theorem 1 we can derive the following estimate of $|\psi|_{\infty}$ for $t \in [t_0,t_1]$

$$|\psi(\mathbf{x},t)| \le \Lambda \left(Q(\Delta_{\psi}) + \|\hat{\boldsymbol{\theta}}(t_0) - \boldsymbol{\theta}\|_{(4D\Gamma)^{-1}}^2 \right)$$

Given that norm $\|\hat{\boldsymbol{\theta}}(t_0) - \boldsymbol{\theta}\|_{(4D\Gamma)^{-1}}^2$ is not increasing, we can bound function $\psi(\mathbf{x}, t)$ for any time moments $t : \sigma_j(t) = 1$ as follows:

$$|\psi(\mathbf{x},t)| \le \Lambda \left(Q(\Delta_{\psi}) + \|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\|_{(4D\Gamma)^{-1}}^2 \right)$$

On the other hand, due to the smoothness of function $\psi(\mathbf{x},t)$ and Assumption 6 one can show that $\psi(\mathbf{x},t)$ is bounded for every $t: \sigma_j(t) = 0$. Hence, as follows from Assumption 1, state \mathbf{x} of the system is bounded. In order to complete the proof we must show that $\hat{\boldsymbol{\theta}}(t) \to \boldsymbol{\theta}$ as $t \to \infty$. We have just shown that state $\mathbf{x}(t)$ is bounded. Then it is bounded for those time intervals when $\sigma_j = 1$ (i.e., when the estimator is turned on). This implies that for any $k = 1, 2, ..., \infty$ the difference $t'_k - t_k > \delta_t > 0$ (i. e., the time when the estimator is

turned on is bounded from below). Therefore, assuming that L is sufficiently small (for instance, $L < \delta_t/2$) and applying the same arguments as in the proof of Theorem 2, we can show that

$$\|\hat{\boldsymbol{\theta}}(t_k') - \boldsymbol{\theta}\| \le \|e^{\left(-\frac{\Gamma D_1 \delta}{L}(n-1)\right)}\|\|\hat{\boldsymbol{\theta}}(t_k) - \boldsymbol{\theta}\|,$$

where $t'_k = t_k + nL + r$, $0 \le r < L$. The corollary is proven.

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